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A finite-size scaling analysis of the localization properties of one-dimensional quasiperiodic systems

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Abstract. We show that the phenomenological renormalization-group analysis based on the finite-size scaling hypothesis is very effective in analysing the critical properties of the localization-delocalization transition (LDT) of the wavefunctions in quasiperiodic systems. We have applied it successfully to the LDT of the discrete Schrödinger equation with an incommensurate modulation potential characterized by a quadratic irrational, showing that different models with a common incommensurate ratio belong to the same universality class if their potential functions are smooth. Moreover, there exists a one-to-one correspondence between the whole universality classes of the LDT and the whole equivalence classes of quadratic irrationals with respect to the modular transformation.

1. Introduction

One-electron properties of a quasiperiodic system are of current interest in connection with quasicrystals (for review, see Sokoloff 1985, Hiramoto and Kohmoto 1992). A simple model is the following one-dimensional Schrödinger equation in the tight-binding approximation, which will be referred to as the discrete Schrödinger equation (DSE):

$$-u_{n-1}-u_{n+1}+V_n u_n = E u_n \tag{1a}$$

$$V_n = Vf(n\omega+\phi) \tag{1b}$$

where $f(x)$ is a periodic function, $f(x+1)=f(x)$, V (>0) the potential strength in units of the transfer integral, ω an irrational number and ϕ the phase variable. The Harper model is a special case where $f(x)=2\cos(2\pi x)$. On the other hand, the potential function of the Fibonacci lattice is an asymmetrical rectangular wave such that its one-period is defined by $f(x)=\theta(x-\omega_G)$, $x\in[0,1]$, where θ stands for the step function and $\omega_G=1/\tau_G$ with $\tau_G=(1+\sqrt{5})/2$ being the golden ratio. It should be noted that $f(x)$ is analytic in the Harper model but not in the Fibonacci lattice. The case where ω is a quadratic irrational is particularly important in connection with the quasicrystal.

We may consider V_n a modulation potential introduced into a periodic lattice. The spatial frequency of the modulation is given by $|\omega|$, while the period by $1/|\omega|$. Since n assumes only integers, two irrationals ω and ω' yield the same modulation potential if $\omega'=m\pm\omega$ with $m\in \mathbf{Z}$.

It has been established that the Harper model exhibits the localization-delocalization transition (LDT) as V is varied (Aubry and André 1980). All the wavefunctions (eigenstates) are extended if $V<1$ but are localized exponentially if $V>1$; the wavefunctions are critical at $V=1$. Correspondingly, the energy spectrum is absolutely continuous, singular continuous or point according as $V<1$, $=1$ or >1, respectively.

The energy spectrum is symmetric on account of the inversion symmetry: $f(x) = -f(x+\frac{1}{2})$. The critical wavefunctions are of multifractal nature (Siebesma and Pietronero 1987, Hiramoto and Kohmoto 1989): their $f(\alpha)$ spectra depend on the energy levels and the incommensurate ratio ω. On the other hand, the wavefunctions in the case of the Fibonacci lattice are known to be critical irrespective of the value of V. They are also multifractals (for the Fibonacci lattice, see the review by Kohmoto 1987).

A critical wavefunction of the Harper model has a hierarchical structure (Thouless and Niu 1983) corresponding to a similar structure of the continued-fraction expansion (CFE) of $\omega = [n_1 n_2 n_3 \ldots]$, which is obtained by the recursive equations: $n_i = [\omega_i]$ and $\omega_i = n_i + 1/\omega_{i+1}$, $i = 1, 2, \ldots$, with $\omega_1 \equiv \omega$ and $[*]$ being the Gaussian symbol. The CFE yields a sequence of best rational approximants to ω. A 'periodicity' in a hierarchical structure gives rise to self-similarity. This is the case where ω is a quadratic irrational, which has a periodic CFE. We shall confine our arguments to this case. Then, ω has a subsequence of best approximants M_1/N_1, $M_2/N_2, \ldots$ such that M_{k+1}/M_k and N_{k+1}/N_k tend to another quadratic irrational τ (>1) which is rationally related to ω. The ratio of self-similarity of the critical wavefunction is equal to τ or some power of it.

The LDT of the DSE was investigated with renormalization-group (RG) analysis by Suslov (1982) and, independently, by Thouless and Niu (1983). Their arguments are, however, not rigorous and assume that n_i in $\omega = [n_1 n_2 n_3 \ldots]$ satisfy $n_i \gg 1$. A more rigorous argument was made by Ostlund and Pandit (1984) on the band centre state but the RG analysis has been implemented only in the case of the Harper model with $\omega = \omega_G$.

The critical behaviour of the LDT will not depend on the microscopic details of the system because the 'correlation length' is divergent at the critical point. Therefore it is interesting to classify different systems into appropriate universality classes, which may depend on ω, the energy levels and, also, on the analyticity of $f(x)$. The aim of the present paper is to investigate this subject by means of numerical analysis.

A numerical method is applicable only to finite systems and we must extract from the data of the systems the properties in the thermodynamical limit. An efficient method for the purpose is the phenomenological RG analysis (Barber and Selke 1982) based on the finite-size scaling hypothesis (Fisher 1971). We shall call it simply the finite-size scaling (FSS) analysis. It has been applied successfully to the Monte Carlo study of phase transitions. In this paper, we shall apply it to the critical properties of the LDT of the DSE.

In section 2 we introduce an order parameter characterizing the LDT and define several critical exponents. Then we examine the FSS hypothesis. We show in section 3 the procedure for the FSS analysis, which we apply to the LDT of the Harper model in section 4 and to the DSE with a smooth potential function in section 5. We classify in section 6 the DSEs with different incommensurate ratios into universality classes with respect to the LDT. In section 7, we discuss briefly remaining subjects on the LDT and, finally, conclude this paper.

2. The order parameter, critical exponents and the FSS hypothesis

We calculate numerically the wavefunctions of finite systems with periodic boundary condition (PBC). To employ the PBC, we must replace ω by its rational approximant

$\tilde{\omega} = M/N$ with M and N being integers. Then the system size is equal to N, which increases geometrically if the sample sequence associated with a suitable sequence of best approximants to ω is used. This property fits the FSS analysis.

Let $u = (u_1, u_2, \ldots, u_N)$ be a normalized eigenvector. Then the 'sum over states' is defined by

$$Z_N(q) = \sum_{n=1}^{N} |u_n|^{2q} \tag{2}$$

with q being a real parameter. The generalized participation ratio is defined with $Z_N(q)$ as

$$l_N(q) = \{Z_N(q)\}^{1/(1-q)}. \tag{3}$$

$l_N(2)$ is the conventional participation ratio, which has been used in the investigation of the Anderson localization in a disordered system (Thouless 1974). It can be shown generally for $q > 0$ that $l_N(q)$ *gives a measure of the extension of the wavefunction in* the localized regime, while it grows linearly with N in the extended regime.

We will digress for a short time to thermodynamic properties of the Ising model. Let M be the instantaneous magnetization of the system composed of N spins. Then the time average of M^2 is written for $T > T_c$ as $\langle M^2\rangle \propto \chi/N$ with χ being the magnetic susceptibility. On the other hand, $\langle M^2\rangle$ tends in the thermodynamic limit, $N \to \infty$, to σ^2 with $\sigma = \langle M\rangle$ being the average magnetization, which is non-vanishing only when $T < T_c$.

It follows that $l_N(q)/N$ has similar properties to $\langle M^2\rangle$ in the Ising model and we may identify the quantity,

$$\sigma_N(q) = \{l_N(q)/N\}^{1/2} \tag{4}$$

with the order parameter of the LDT. The thermodynamic limits of $l_N(q)$ and $\sigma_N(q)$ are denoted as $l(q)$ $(\equiv l_\infty(q))$ and $\sigma(q)$ $(\equiv \sigma_\infty(q))$, respectively. $l(q)$ corresponds to the magnetic susceptibility of the Ising model. $l(q)$ (or $\sigma(q)$) is non-trivial only in the localized (or extended) regime.

We shall confine our arguments to the ground-state wavefunction of the DSE and assume that it exhibits a LDT at $V = V_c$; $V < V_c$ or $V > V_c$ is the extended or localized regime, respectively. The three critical exponents are defined as in the case of the Ising model:

$$\xi \sim |v|^{-\nu} \qquad l(q) \sim v^{-\gamma} \qquad \text{and} \qquad \sigma(q) \sim (-v)^{\beta} \tag{5}$$

where ξ is the correlation length and $v = (V - V_c)/V_c$. ξ is identical to the localization length in the localized regime (Siebesma and Pietronero 1987). Note that β and γ are dependent on q, $\beta = \beta(q)$ and $\gamma = \gamma(q)$, while ν is not.

Our arguments in the rest of this section are not restricted to the one-dimensional system. On the basis of a similar consideration to that in statistical mechanics (Fisher 1971), we assume the following FSS relationship for a finite but large system:

$$\sigma_N(q)^2 L^{d-\gamma/\nu} = F(L^{1/\nu} v) \tag{6}$$

where d stands for the dimensionality of the system and L the linear dimension; $N = L^d$. Then the size dependence of $\sigma_N(q)$ obeys a power law at the critical point, $v = 0$ $(V = V_c)$: $\sigma_N^2(q) = F(0)L^{\gamma/\nu - d}$ and, consequently, $l_N(q) = F(0)L^{\gamma/\nu}$. We shall call $F(x)$ the scaling function.

The behaviour of $F(x)$ in the limit $x \to \infty$ (or $-\infty$) describes the property of $l(q)$ (or $\sigma(q)$) in the localized (or extended) regime, so that $F(x)$ must obey asymptotically the power law $x^{-\gamma}$ (or $(-x)^{\nu d-\gamma}$). Therefore we can conclude the following hyper-scaling law for the exponents:

$$\frac{2\beta}{\nu}+\frac{\gamma}{\nu}=d. \tag{7}$$

That is, only two of the three exponents are independent of each other. Note that β/ν and γ/ν are the exponents associated with the relationships of $\sigma(q)$ and $l(q)$ to ξ, respectively; $\sigma\sim\xi^{\beta/\nu}$ and $l\sim\xi^{\gamma/\nu}$.

We will refer briefly to the multifractal property of the critical wavefunction (Siebesma and Pietronero 1987, Hiramoto and Kohmoto 1989). Let $S(q)$ be the thermodynamic limit of $\log(Z_N(q))/\log N$ at $V=V_c$. Then it is related to the generalized fractal dimension D_q (Halsey *et al* 1986) of the critical wavefunction by $S(q)\equiv(1-q)D_q$, which together with equation (3) and $l_N(q)=F(0)L^{\gamma/\nu}$ yields $D_q=\gamma(q)/\nu$. The $f(\alpha)$ spectrum characterizing the multifractal property of the critical wavefunction is given as the Legendre transform of $S(q)$. It follows that the critical properties of the LDT are dominated by the fractal properties of the critical wavefunction. Note that D_q depends on q, in contrast to the case of a simple fractal.

3. The FSS analysis

Let us define a function of two size-variables by

$$R[N, N']\equiv\frac{\log\{\sigma_N(q)^2/\sigma_{N'}(q)^2\}}{\log(L/L')}+d. \tag{8}$$

Then it takes γ/ν at $V=V_c$ for any pair $\{N, N'\}$ provided that N and N' are sufficiently large. Therefore we can determine γ/ν and V_c as the ordinate and the abscissa of the common crossing point (the fixed point) of the plots of $R[N, N']$ versus V for different pairs of N and N'. This is the essence of the FSS analysis (Barber and Selke 1982).

On the other hand, the exponent ν can be determined by the condition that the plot of $y\equiv\sigma_N^2(q)L^{d-\gamma/\nu}$ versus $x\equiv L^{1/\nu}v$ is represented by a single curve, $y=F(x)$, for different values of N. This plot shall be referred to as the FSS plot. In order to see the asymptotic behaviour of $F(x)$ in the limit $x\to\pm\infty$, it is convenient to investigate the $\log y$ versus $\log|x|$ plot; it will exhibit a linear variation in the region $0<\log|x|<\infty$ but will be nearly constant ($=\log F(0)$) in the region $-\infty<\log|x|<0$. The slope in the asymptotic regime $x\to\infty$ (or $-\infty$) is equal to $-\gamma$ (or 2β). This change of behaviour from the constant regime to the linear-variation one shows the crossover from the critical regime, $\xi\gg L$, to the thermodynamic one, $\xi\ll L$.

The above-mentioned crossover can be observed in the $\log\sigma_N(q)$ versus $\log L$ plot as well. The abscissa of the crossover point is approximately equal to $\log\xi$, so that it varies with V. This allows us to determine the dependence of ξ on V, which was used by Siebesma and Pietronero (1987) in their numerical investigation of $\xi=\xi(V)$.

4. Application of the FSS analysis to the Harper model

We shall assess the effectiveness of the FSS analysis by applying it to the Harper model, for which several exact results are known. We assume $\omega=\omega_G$ $(=1/\tau_G)$. It is well known that $\tau_G=[111\ldots]$ and a best approximant to ω_G is written as F_{k-1}/F_k, with F_k being

Fibonacci numbers: F_k are generated by the recursion relation $F_{k+1} = F_k + F_{k-1}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Note that $\lim F_{k+1}/F_k = \tau_G$. The Fibonacci sequence, $\{F_k\}$, has a three-cycle parity sequence, $\{+--+--+\ldots\}$. Consequently, the sequence $\{\sigma_N(q)|N = F_k,\ k = 1, 2, \ldots\}$ is divided into three subsequences with different scaling functions. The growth rate τ of the size N in each subsequence is given by $(\tau_G)^3$ $(=2+\sqrt{5} = \lim F_{k+3}/F_k)$.

We plot $\sigma_N(2)$ versus V in figure 1 for $N = F_{3i}$, $i = 4$–8. The size dependence of $\sigma_N(2)$ is quite similar to a plot of $(\langle M^2 \rangle)^{1/2}$ in the case of the Ising model. $\sigma_N(2)$ tends rapidly to $\sigma(2)$ as V falls away from the critical point while it decreases in proportion to $N^{-1/2}$ as V goes beyond the critical point.

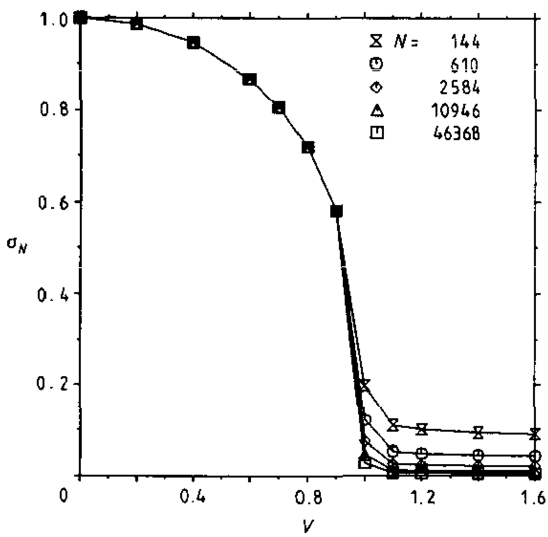

Figure 1. The plots of $\sigma_N(2)$ versus V for samples with different sizes. We can observe that $\sigma_N(2)$ will converge in the thermodynamic limit to $\sigma(2)$.

We show in figure 2 $R[N, N']$ versus V in the vicinity of $V = V_c$. From this plot we have determined V_c, yielding $V_c = 1$ to six digits, in agreement with the exact result. The critical exponent γ/ν has been determined to be 0.329, where the next digit has been rounded considering the numerical error. The accuracy of the exponent is lower than that of V_c because a small error in V_c gives rise to a large error in the exponent.

The FSS plot is shown on a linear scale in figure 3(*a*) and on a logarithmic scale in figure 3(*b*). The FSS plot in figure 3(*a*) scatters least when ν is chosen to be 1.00. Only the data for $N = 2{,}584$ and 46,368 are plotted in figure 3(*b*), the log-log plot. We have confirmed that the data for $N = 10{,}946$ are in good agreement with these data. However, the data for $N = 144$ and 610 deviate slightly in the region $|v| > 0.05$, which may be accounted for by introducing a correction term to the FSS relation. In any case, the FSS plot confirms the FSS hypothesis formulated by equation (6). The three critical exponents have been determined as $\nu = 1.00$, $\gamma = 0.329$ and $\beta = 0.336$.

Aubry and André (1980) have obtained an exact expression for ξ in the localized regime: $\xi = 1/\log V$, $V > 1$. On the other hand, Siebesma and Pietronero (1987) obtained it in the extended regime by a numerical investigation: $\xi = 1/|\log V|$, $V < 1$. These results yield $\xi = 1/|v|$ for $|v| \ll 1$, so that $\nu = 1$. Thus, our calculation confirms these results.

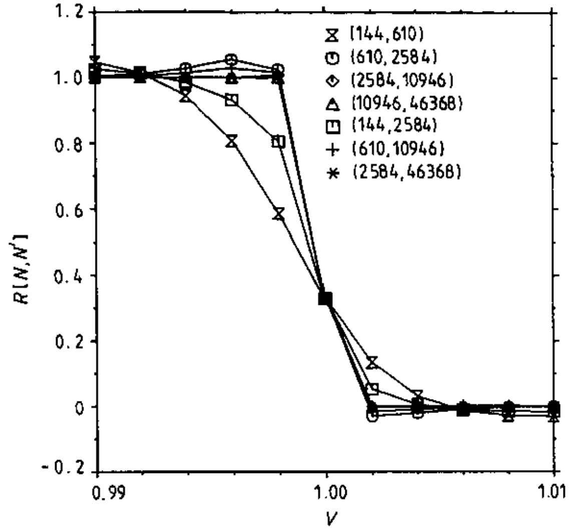

Figure 2. The plots of $R[N, N']$ versus V for several pairs of N and N'. All the plots cross at one point, whose abscissa and ordinate give the critical potential strength V_c (=1.00) and the critical exponent γ/ν (=0.329).

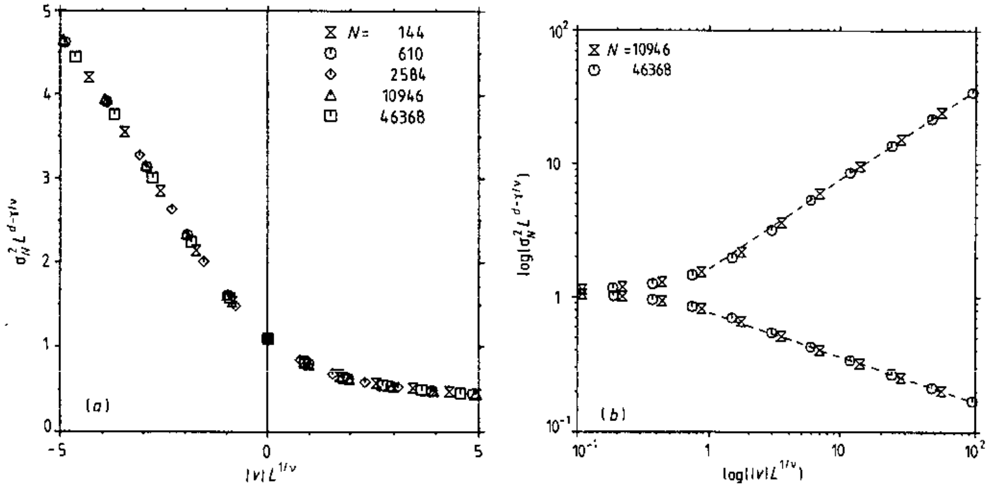

Figure 3. The FSS plot in the linear scale (a) and the logarithmic scale (b). In (a) the data for different sizes are represented best by a single curve when ν is chosen to be 1.00. Note that $d=1$ and $L=N$. In (b) the abscissa stands for $\log(|v|L^{1/\nu})$, so that the scaling function yields two branches; the upper or the lower branch refers to $v<0$ or $v>0$, respectively. Two broken lines show the power laws in the asymptotic regime $|v|L^{1/\nu} \gg 1$. Their slopes are given by 2β and $-\gamma$. On the other hand, the asymptote in the limit $\log(|v|L^{1/\nu}) \to -\infty$ (or, equivalently, $|v|L^{1/\nu} \to 0$) is a horizontal line whose height is equal to $\log F(0)$.

We have calculated $S(q)$ as a function of q, showing a nonlinear dependence on q. The resulting $f(\alpha)$ spectrum was in good agreement with figure 6(a) in Hiramoto and Kohmoto (1989) and is not reproduced here. The critical wavefunction obtained numerically is shown in figure 4(a). The wavefunction appears to scale with τ_G but the exact scale is found to be $(\tau_G)^3$, which is consistent with the three-cycle nature of the sequence.

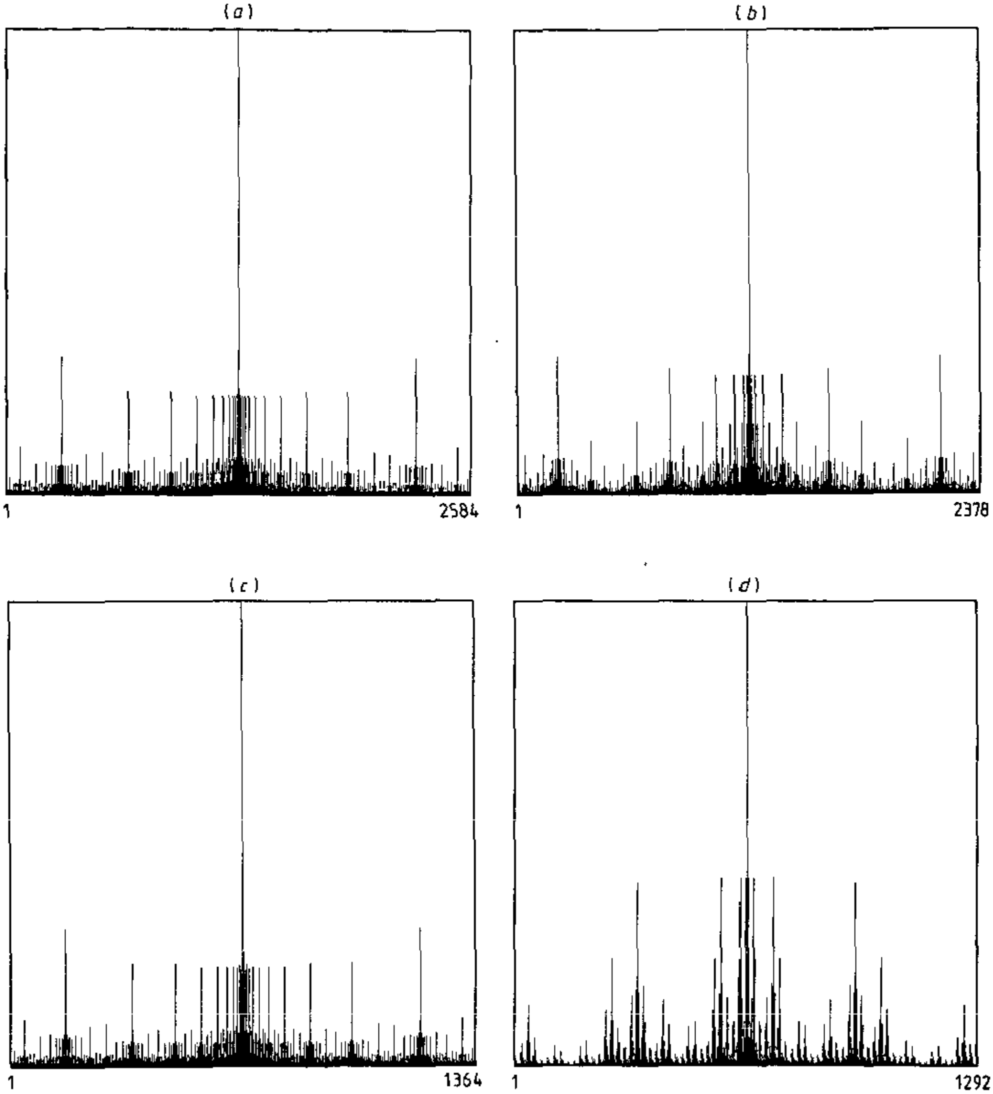

Figure 4. The ground state wavefunction of the Harper models at the critical point $V=1.0$ for (*a*) $\omega=\omega_G\,(\equiv 1/\tau_G)$, (*b*) $\omega=1/\tau_S$, (*c*) $\omega=\omega_G/\sqrt{5}$ and (*d*) $\omega=2\omega_G$. Each wavefunction is obtained from a finite system with size N by assuming the PBC. The sites of the system are numbered from 1 to N. ω_G and $\omega_G/\sqrt{5}$ are M-equivalent and the wavefunctions (*a*) and (*c*) are hardly distinguished except for the scales of the abscissa. Note that $1364/2584 \approx \sqrt{5}/(\tau_G)^3$. The wavefunctions are approximately self-similar with the ratios τ_G, τ_S, τ_G and $2+\sqrt{5}$, respectively. The peak heights of the outermost satellites in (*a*) deviate appreciably from others on account of the PBC.

5. The DSEs with smooth potential functions

The potential function $f(x)$ is expanded into a Fourier series as

$$f(x)=a_0+\sum_{k=1}^{\infty} a_k \cos(2\pi kx) \tag{9}$$

where we have assumed for brevity that $f(x)$ is even. Using equation (9) we can expand the modulation potential V_n into 'plane waves', whose wavenumbers are given by $k\omega$ with $k \in Z$.

The energy spectrum of the Harper model is pure on account of its 'self-duality' (Aubry and Andre 1980). An absolutely continuous spectrum and a point spectrum can coexist at some potential strength in other models (Suslov 1982, Soukoulis and Economou 1982, Hiramoto and Kohmoto 1989).

We shall investigate the LDT of the ground state of the DSE with a smooth potential function. We take four models with inversion-symmetric potential functions $f_i(x)$, $i=1$–4, for which the Fourier component a_n with even n vanishes. The first two have only two harmonics with (1) $a_1=1$ and $a_3=\frac{1}{2}$ and (2) $a_1=\frac{1}{2}$ and $a_3=1$. The third model is a periodic superposition of Gaussian functions:

$$f_3(x)=\sum_{m=-\infty}^{+\infty}(-1)^m \exp(-\alpha(x-m/2)^2) \tag{10}$$

whose Fourier coefficients are rapidly decreasing: $a_{2m+1} \propto \exp(-\pi^2(2m+1)^2/\alpha)$. The parameter α is chosen to be $8\pi^2/\log 2$ so that $a_3/a_1=\frac{1}{2}$. The fourth model is given by equation (9) with $a_{2m+1}=(-1)^m/(2m+1)^3$, $m=0,1,2,\ldots$ $f_i(x)$, $i=1$-3, are analytic but $f_4(x)$ is only smooth (the C^1 class). In fact, $f_4(x)$ is a 'parabolic wave', which is a smooth and periodic junction of convex parabolas and concave ones. In order to compare with the Harper model, we shall normalize the potential functions of these models so that they satisfy $f_{\max}=-f_{\min}=2$.

We have implemented the FSS analysis for these four models with $\omega=\omega_G$ and determined V_c and the critical exponents. The results for V_c of the models 1–4 are 0.539 997, 0.445 009, 0.496 614, and 1.086 734, respectively, which are all different from V_c $(=1)$ of the Harper model; *V_c has a tendency to have a smaller value if the curvature of $f(x)$ at its minimum is larger.* In contrast, the critical exponents ν and $\gamma(2)$ are found all to be in agreement with those of the Harper model within the numerical accuracy which is better than 0.001. We have found also that the critical wavefunctions of these models are hardly distinguished from figure 4(*a*) by inspection. Therefore, the Harper model and the present four models belong to a single universality class with respect to the LDT.

We have implemented the FSS analysis for the fifth model, which has the following analytic potential function:

$$f_5(x)=\sum_{m=-\infty}^{+\infty}\exp(-\alpha(x-m)^2). \tag{11}$$

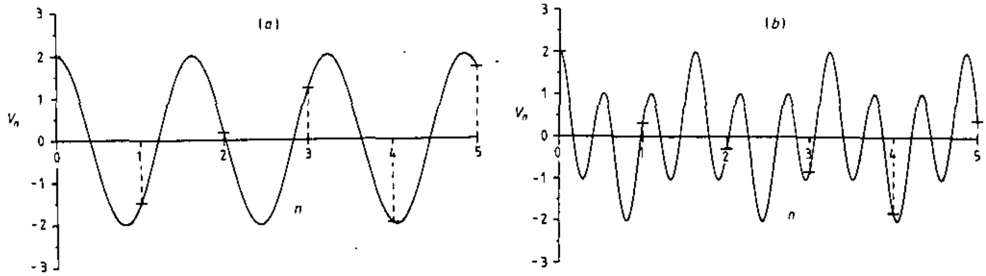

Figure 5. A comparison of the potential function of the Harper model (*a*) and a model including third-order harmonics (*b*). The modulation period is taken to be τ_G. The values of the potential at the lattice points are shown by horizontal bars.

This is not inversion-symmetric. This model has been found to belong to the same universality class as that of the Harper model. Thus, we can conclude that the DSEs with smooth potential functions and a common ω all belong to a single universality class. We should note in this respect that Suslov (1982) made a persuasive argument, though not rigorous, that ν is universally equal to 1.

The potential function $f_2(x)$ is compared in figure 5 with that of the Harper model. It is remarkable that the two DSEs belong to the same universality class regardless of the large difference in the potential function.

6. The universality classes of the LDT of the DSE with a quadratic irrational as its incommensurate ratio

We confine our arguments to the case of the Harper model because the higher-order harmonics in the potential function are indifferent to the critical properties of the LDT. We consider first the case $\omega = 1/\tau_S$ with $\tau_S = 1+\sqrt{2}$ (=[222...]) being the silver ratio. The critical exponent $\gamma(2)$ has been determined to be 0.340, so that this model belongs to a different universality class from that of the case of $\omega = \omega_G$. The critical wavefunction is shown in figure 4(*b*). It has a self-similarity with the scale $(\tau_S)^2$ and its profile is markedly different from figure 4(*a*).

A later term of the CFE of ω is related to a long-range behaviour of the critical wavefunction. Therefore, it is expected that two Harper models with ratios ω and ω' belong to the same universality class if ω and ω' have a common tail in their CFEs. This is expected also from the renormalization-group structure of the LDT (Suslov 1982, Thouless and Niu 1983, Ostlund and Pandit 1984). A necessary and sufficient condition for two quadratic irrationals ω and ω' to have a common tail is that they are related by the modular transformation (Hardy and Wright 1938)

$$\omega' = \frac{a\omega + b}{c\omega + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \boldsymbol{Z}). \tag{12}$$

If this relation is satisfied, we shall say ω and ω' are M-equivalent. Since GL(2, $\boldsymbol{Z}$) is a group, the set of all the quadratic irrationals are grouped into equivalence classes which are disjoint. Thus we have arrived at the conjecture that an equivalence class of quadratic irrationals corresponds to a universality class of the LDT.

The case $\omega = 1/(1+\tau_G)$ has the same period as that of ω_G because $\omega = 1-\omega_G$ but the case $\omega = 1/(2+\tau_G)$ $(=\omega_G/\sqrt{5})$ has a different period. The latter ω is M-equivalent to ω_G since $\omega = [03111\ldots]$. The critical wavefunction for this ratio is similar (as shown in figure 4(*c*)) to that in the case of $\omega = \omega_G$. On the other hand, $2\omega_G$ (=[3444...]), the second harmonic to ω_G, is not M-equivalent to ω_G and the critical wavefunction in this case is markedly different from that of ω_G as shown in figure 4(*d*); the ratio of self-similarity is equal to $(\tau)^2$ with $\tau = 2+\sqrt{5}$ (=[444...]). We have investigated several non-trivial pairs of M-equivalent quadratic irrationals and confirmed that the critical wavefunctions are similar in each pair, so that the above-mentioned conjecture has been verified numerically.

We should note at this point that this result is quite similar to the one which has been established in the theory of the critical property of the circle map with respect to the transition from quasiperiodicity to chaos (Ostlund *et al* 1983).

7. Discussions and conclusions

We have concentrated up to this point on the ground-state wavefunction of the DSE. The DSE with an inversion-symmetric potential function has an eigenstate at $E=0$, the band centre. We have implemented the FSS analysis for the LDT of the band-centre wavefunction and obtained similar results on the universality to the case of the ground state. The critical exponent ν of the centre state has been always equal to 1. On the other hand, $\gamma(2)$ has a different value from that of the ground state of the same model. For example, $\gamma(2)=0.612$ when $\omega=\omega_G$. The same value was obtained for this case by Evangelou and Economou (1991). Our result is consistent with the fact that the $f(\alpha)$ spectrum is different for the ground state and the centre state (Hiramoto and Kohmoto 1989). These results show that the ground state and the centre state belong to different universality classes with respect to the LDT.

The energy spectrum of the DSE has an infinite number of gaps and the ground state (or the centre state) is a representative of the edge states (or the centre states) of the sub-bands. All the edge states (or the centre states) have a common $f(\alpha)$ spectrum in the case of the Harper model (Hiramoto and Kohmoto 1989) and there are no reasons why this does not hold in the case of the DSE with a smooth potential function.

We have concentrated our considerations on models with smooth potential functions, which we shall call smooth models. We consider briefly non-smooth models. A representative of them is the Fibonacci lattice. It exhibits a remarkably different behaviour from the Harper model with respect to the localization properties (Hiramoto and Kohmoto 1992), so that it cannot be incorporated in any universality class including smooth models.

The potential function of the Fibonacci lattice is discontinuous and we should consider next a model with a continuous but non-smooth potential function. A simple model with this property is the one whose potential function is a triangular wave. We have performed the FSS analysis for the ground-state wavefunction of this model and found that $V_c=0$. That is, the wavefunction is localized unless $V=0$. Moreover, we have found that the growth of ξ and l when V (>0) tends to 0 does not obey the power law but the exponential law, $\xi, l \sim \exp(c/V)$. This means that $V=0$ is an essential singularity. On account of the exponential law, we have to redefine v as $v=\exp(-1/V)$, where v is the variable appearing in equations (5) and (6). The FSS plot of the present model is shown in figure 6. The exponent γ/ν has been found to be equal to 1, which is consistent with the fact that 'the critical wavefunction' of this model is not a fractal but a simple Bloch state ($u_n=1/\sqrt{N}$).

The scaling functions of thermodynamic systems belonging to a single universality class are believed to be universal provided that the relevant variables are normalized appropriately. Moreover, there exists one universal amplitude ratio for each scaling law among the critical exponents (Aharony and Hohenberg 1976). We can expect that to be true also for the LDT of quasiperiodic systems. This subject will be discussed elsewhere.

The FSS analysis is quite versatile, in contrast to the analytical method. It is not difficult, for example, to apply it to higher dimensional quasiperiodic systems and we are starting research along this line.

In conclusion, we have successfully applied the FSS analysis to the LDT of the DSE with a smooth potential function and an incommensurate ratio being a quadratic irrational and found that there exists a one-to-one correspondence between the whole

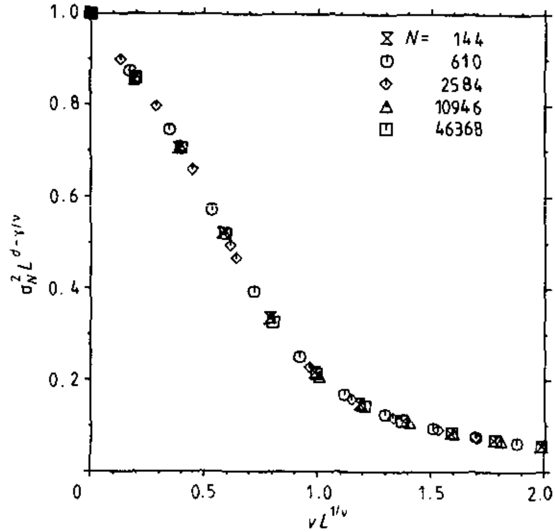

Figure 6. The FSS plot of the model with the potential function of the triangular wave. The variable v in the abscissa is defined as $v = \exp(-1/V)$. The 'critical exponent' ν in $\xi \sim v^{-\nu}$ is determined as 1.84, while $\gamma/\nu = 1.00$.

universality classes of the LDT and the whole equivalence classes of quadratic irrationals with respect to the modular transformation.

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